Journal of New Frontiers in Mathematics and Statistics

ISSN 2719-2415 (Online)

Volume 4, Issue 1, 2025, 15-45

Home page: www.imathm.edu.lk/publications

Block Second Derivative Methods for Solving Financial Chaotic Models

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ABSTRACT

The paper presents a new numerical method, called second derivative modified generalized backward differentiation formulae blended with backward differentiation formulae (SDMGBDF blended with BDF) with variable step sizes, for solving financial chaotic models. Unlike existing methods that linearize and subdivide the problem, this approach solves chaotic systems directly without linearization or subdivision. The method uses a multistep inversion technique and blends two linear multistep methods. The paper discusses convergence analyses and finds that the proposed block methods are efficient and accurate, making them suitable for solving chaotic initial value problems (IVPs) of ordinary differential equations (ODEs).

Keywords: Block, Second derivative, Financial, Chaotic, Models.

ISSN 2719-2415 (Online)

1. INTRODUCTION

Nonlinear dynamics has emerged as a crucial field of science due to its significance in everyday activities. Mathematical models of these activities are formulated to facilitate understanding of physical phenomena, often resulting in differential equations. These differential equations can be categorized into linear and nonlinear ordinary differential equations (ODEs). However, many problems cannot be easily solved analytically, necessitating the development of numerical methods.

Financial chaotic systems are complex and dynamic, exhibiting nonlinear behaviour that makes predictions and solving challenging. These systems are characterized by unpredictability, non-linearity, interconnectedness, and self-similarity. Understanding financial chaotic systems is crucial for identifying potential risks and opportunities, and for developing effective strategies to navigate these complexes.

This study aims to contribute to the existing literature on financial chaotic systems by developing a novel approach to solve these complex systems. The proposed method has the potential to provide more accurate predictions and approximation, which can help financial institutions and policymakers make informed decisions.

2. LITERATURE REVIEW

Numerical methods for solving chaotic initial value problems (IVPs) of ODEs have been extensively studied. Chaotic systems exhibits complex behaviour characterized by sensitivity to initial conditions, topological transitivity, and dense periodic orbits (Davaney, 2003; Basener, 2006). Semi-analytic methods, such as Adomian decomposition and multistage differential transformation, have been used to solve chaotic ODEs (Nour et al., 2012). However, these methods lack global convergence, are tedious, and introduce errors. Recent studies have focused on developing new block second derivative linear multistep methods (LMMs) with improved stability and accuracy. Blended second derivative LMMs are a class of numerical methods used to solve ordinary differential equations (ODEs). These methods combine the advantages of different LMMs to produce a more accurate and efficient method. The development of blended second derivative LMMs dates back to the work of Brugnano & Trigiante (1998), who introduced the concept of blending different LMMs to produce a more accurate method. Since then, several researchers have contributed to the development of blended second derivative LMMs (Omar et al., 2021). Blended second derivative LMMs have several desirable properties, including: High accuracy, blended second derivative LMMs can achieve high accuracy by combining the advantages of different LMMs (Brugnano & Trigiante, 1998), stable and efficient. Blended second derivative LMMs have been compared with other methods, Runge-Kutta methods: Blended second derivative LMMs have been shown to be more accurate and efficient than Runge-Kutta methods for solving certain problems (Omar et al., 2021) and Adams-Bashforth methods.

ISSN 2719-2415 (Online)

The motivation for this research stems from the need to develop accurate and efficient numerical methods for solving chaotic systems of ODEs. Chaotic systems are ubiquitous in nature and are used to model complex phenomena in field such as physics, biology, and finance.

There has been a flurry of research activities in finding accurate numerical and analytical methods for solving chaotic systems of ODEs. Chaotic ODEs have complicated dynamical systems, nonlinear aperiodic oscillators which are highly sensitive to small changes in the initial conditions with their solutions rapidly changing from being stable to unstable and non-periodic long-term behaviour. Solving these chaotic systems has been and still remains a challenge to researchers. Due to this reasons, the study of chaotic systems has increased because of their application in various areas. Most often, chaotic ODEs cannot be solved analytically; numerical methods have to be used. Many researchers used the classical numerical methods to solve the chaotic systems. Lorenz (1972) used the explicit Euler scheme with the central-difference scheme, Yao (2010) and Yorke & Yorke (1979) used the Adams method, Sparrow (1982) used the higher derivative scheme and Sarra & Meador (2011) used the Runge -Kutta method. Sandile, Vusi, & Precious (2017) used the multi-domain spectral relaxation method to solve chaotic ODEs. The interval of integration was discredited over non-overlapping sub-intervals of the domain. Also, the afore-mentioned and others used continuity conditions to advance the solution across the non-overlapping sub-intervals. The setback of this method is that it involves linearization of the nonlinear part and solving over multiple domains before taking the union, which introduces errors.

These methods might not give correct solution due to the instability properties over a given time interval, since the global error grows as time increase if a large interval is considered; this is noted in Lorenz’s (1972) (butterfly effect).

However, existing methods have limitations, such as lack of global convergence, tedious computations, time step selection, and introduction of errors. In this study, these challenges are tackled using the multistep collocation approach to derive the continuous conventional methods of second derivative block Linear Multistep Methods (LMMs) with variable step sizes.

The following research questions are considered: Can the conventional second derivative LMMs with variable step size be able to solve nonlinear IVPs of ODEs with chaotic properties? Are the regions of absolute stability of the conventional second derivative LMMs stable? Is the numerical procedure involved in computing solutions tedious and time-consuming when the variable step sizes? Can global convergence be achieved using longer intervals without splitting?

The primary objective of this study is to developed novel approach to solve financial chaotic systems using second derivative linear multistep methods. The specific objectives are: To derive a new formula by combing two different methods, enhancing the stability and accuracy of the resulting formula. To evaluate the effectiveness and efficiency of the proposed method in solving existing real-life financial chaotic ODEs.

ISSN 2719-2415 (Online)

To compare the numerical solutions, trajectories, phase space and phase portraits of the proposed method with existing methods used to solve financial chaotic systems and the in-built MATLAB ode 23s.
This study aims to contribute to the existing literature on financial chaotic systems by developing a novel approach to solve these complex systems. The proposed method has the potential to provide more accurate predictions and approximation, which can help financial institutions and policymakers make informed decisions.

Consider, chaotic system of initial value problem defined as

$$y'(x) = f(x, y(x)), y(x_0) = y_0 \tag{1}$$

on the interval $I = [x_0, x_N]$ where $y : [x_0, x_N] \to \Re^m$ and $f : [x_0, x_N] \to \Re^m \times \Re^m$ is continuous and differentiable we are concern with finding the numerical solutions of (1) using second derivative methods. The first section has the introduction to the study, including the background, research motivation, and objectives and literature review; the second section is the material and methods. The third section includes, the convergence and stability analysis while the last section we test the robustness of these new methods by solving some existing real-life financial chaotic ordinary differential equations (ODEs).

Theorem (Existence and Uniqueness of Solutions)
Let f(x, y) be defined and continuous for all points (x, y) in an open region of two-dimensional real Euclidean space D, defined by $a \le x \le b, -\infty < y < \infty,$ a and b finite, and let there exist a constant L such that, for every x, x, y such that (x, y) and (x, y^*) are both in D, $|f(x, y) - f(x, y^*)| \le L|y - y^*|$. Then, if y_0 is any given number, there exist a unique solution y(x) of the IVP (1), where y(x) is continuous and differentiable for all (x, y) in D.

3. MATERIAL AND METHODS

3.1 The Second Derivative Multistep Collocation Method
The method carried out in (Onumanyi et al., 1999), shall be used in this derivation, to construct a k-step second derivative multistep collocation method as

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h_n \sum_{j=0}^{m-1} \beta_j(x) f_{n+j} + h_n^2 \sum_{j=0}^{m-1} \lambda_j(x) y''_{n+j} \tag{2}$$

where h_n is the variable step size, $\alpha_j(x), \beta_j(x),$ and $\lambda_j(x)$ are the continuous of the coefficients method defined as;

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$$\alpha_j(x) = \sum_{i=0}^{t+m-1} \alpha_{j,i+1} x^i,\ j \in \{0,\ldots,t-1\}$$
$$\beta_j(x) = \sum_{i=0}^{t+m-1} \beta_{j,i+1} x^i,\ j \in \{0,\ldots,m-1\} \tag{3}$$
$$\lambda_j(x) = \sum_{i=0}^{t+m-1} \lambda_{j,i+1} x^i,\ j \in \{0,\ldots,m-1\}$$

To determine the continuous coefficients $\alpha_j(x), \beta_j(x),$ and $\lambda_j(x)$ the following conditions are imposed:

$$\begin{aligned} &\alpha_j\left(x_{n+1}\right) = \delta_{ij},\ j = 0,\ldots,t-1; i = 0,\ldots,t-1 \\ &h_n \beta_j\left(x_{n+1}\right) = 0,\ j = 0,\ldots,m-1; i = 0,\ldots,t-1 \\ &h_n^2 \lambda_j\left(x_{n+1}\right) = 0,\ j = 0,\ldots,m-1; i = 0,1,\ldots,t-1 \end{aligned} \tag{4}$$

$$\begin{aligned} &\alpha_j^{'}\left(x_i\right) = 0,\ j = 0,\ldots,t-1; i = 0,\ldots,m-1 \\ &h_n \beta_j^{'}\left(x_i\right) = \delta_{ij},\ j = 0,\ldots,m-1; i = 0,\ldots,m-1 \\ &h_n^2 \lambda_j^{'}\left(x_i\right) = 0,\ j = 0,\ldots,m-1; i = 0,1,\ldots,m-1 \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\alpha_j^{"}\left(x_i\right) = 0,\ j = 0,\ldots,t-1; i = 0,\ldots,m-1 \\ &h_n \beta_j^{"}\left(x_i\right) = 0,\ j = 0,\ldots,m-1; i = 0,\ldots,m-1 \\ &h_n^2 \lambda_j^{"}\left(x_i\right) = \delta_{ij},\ j = 0,\ldots,m-1; i = 0,1,\ldots,m-1 \end{aligned} \tag{6}$$

where x_j, j = 0, 1, …, m-1 are the m distinct collocation points used and $t, 0 < t \le k$ the number of interpolation points and (4) - (5) can be written in matrix form as

$$DC = 1 \tag{7}$$

where I is the identity matrix of dimension $(t+m)\times(t+m)$ while D and C are matrices defined as

ISSN 2719-2415 (Online)

$$D=\begin{pmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_0 & \cdots & (t+m-1)x_0^{t+m-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 1 & 2x_{m-1} & \cdots & (t+m-1)x_{m-1}^{t+m-2} \\ 0 & 0 & 2 & \cdots & (t+m-2)(t+m-1)x_{m-2}^{t+m-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 2 & \cdots & (t+m-2)(t+m-1)x_{m-1}^{t+m-3} \end{pmatrix} \qquad (8)$$

$$C=\begin{pmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{t-1,1} & h_n\beta_{0,1} & \cdots & h_n\beta_{m-1,1} & h_n^2\lambda_{0,1} & \cdots & h_n^2\lambda_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h_n\beta_{0,2} & \cdots & h_n\beta_{m-1,2} & h_n^2\lambda_{0,2} & \cdots & h_n^2\lambda_{m-1,2} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h_n\beta_{0,t+m} & \cdots & h_n\beta_{m-1,t+m} & h_n^2\lambda_{m-1,t+m} & \cdots & h_n^2\lambda_{m-1,t+m} \end{pmatrix} \qquad (9)$$

It follows from (7) the columns of C = D^{-1}, give the elements of the continuous coefficients $\alpha_j(x), \beta_j(x)$, and $\lambda_j(x)$ of the continuous scheme (3).

3.2 Derivation of the Continuous Second Derivative Methods for k = 2 and 3
The Modified Generalized Backward Differentiation Formulae blended with Backward Differentiation Formulae is defined as:

$$y(x)=\sum_{j=0}^{k}\alpha_j(x)y_{n+j}+h\beta_v(x)f_{n+v}+h_n^2\lambda_k(x)f_{n+k} \qquad (10)$$

where, v = $\left[\!\left[\frac{k+1}{2}\right]\!\right]$, and $[\![\quad]\!]$ is the greatest integer function. (11)

Case k =2: v = 2 as defined in (11) and (10) becomes

$$y(x)=\sum_{j=0}^{2}\alpha_j(x)y_{n+j}+h\beta_2(x)f_{n+2}+h_n^2\lambda_2(x)f_{n+2} \qquad (12)$$

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$$D = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 \\ 0 & 0 & 2 & 6x_{n+2} & 12x_{n+2}^2 \end{bmatrix}, \tag{13}$$

Inverting the D matrix in (13) using maple software yields the elements of the matrix C. Substituting the columns of the matrix C in (12) gives the continuous coefficients of the method with $\eta = (\mathrm{x} - x_n)$ and $x = (\eta + x_n)$ as follows:

$$\alpha_o(\eta + x_n) = \frac{1}{8h_n^4}[\eta^4 - 7h_n\eta^3 + 18\,h_n^2\,\eta^2 - 20\,h_n^3\,\eta + 8\,h_n^4]$$

$$\alpha_1(\eta + x_n) = \frac{1}{8h_n^4}[-\eta^4 + 6h_n\eta^3 - 12\,h_n^2\,\eta^2 + 8\,h_n^3\,\eta]$$

$$\alpha_2(\eta + x_n) = \frac{1}{8h_n^4}[7\eta^4 - 41\,h_n\,\eta^3 + 78\,h_n^2\,\eta^2 - 4] \tag{14}$$

$$\beta_2(\eta + x_n) = \frac{1}{4h_n^4}[-3\eta^4 + 17\,h_n\,\eta^3 - 30\,h_n^2\,\eta^2 + 16\,h_n^3\,\eta]$$

$$\lambda_2(\eta + x_n) = \frac{1}{4h_n^8}[\eta^4 - 5\eta^3 + 8\,h_n\,\eta^2 + 4\,h_n^2\,\eta]$$

Substituting the continuous coefficients (14) into (12) produces the continuous interpolant of our new method as:

$$y(\eta) = \frac{1}{8h_n^4}[\eta^4 - 7h_n\eta^3 + 18\,h_n^2\,\eta^2 - 20\,h_n^3\,\eta + 8\,h_n^4]y_n + \frac{1}{8h_n^4}[-\eta^4 + 6h_n\eta^3 - 12\,h_n^2\,\eta^2 + 8\,h_n^3\,\eta]y_{n+1} + \frac{1}{8h_n^4}[7\eta^4 - 41\,h_n\,\eta^3 + 78\,h_n^2\,\eta^2 - 4]y_{n+2} + \frac{1}{4h_n^4}[-3\eta^4 + 17\,h_n\,\eta^3 - 30\,h_n^2\,\eta^2 + 16\,h_n^3\,\eta]f_{n+2} + \frac{1}{4h_n^8}[\eta^4 - 5\eta^3 + 8\,h_n\,\eta^2 + 4\,h_n^2\,\eta]g_{n+2} \tag{15}$$

The differentiating the continuous interpolant once and evaluating at $\eta = \{0, h_n\}$, yields the following discrete schemes:

$$\boldsymbol{y_{n+1} = \frac{1}{8}h_n^2 g_{n+2} - \frac{5}{8}h_n f_{n+2} - \frac{1}{2}h_n f_{n+1} + \frac{17}{16}y_{n+2} + \frac{1}{16}y_n}$$
$$\boldsymbol{y_{n+2} = -\frac{2}{17}h_n^2 g_{n+2} + \frac{10}{17}h_n f_{n+2} + \frac{8}{17}h_n f_{n+1} + \frac{16}{17}y_{n+1} + \frac{1}{17}y_n} \tag{16}$$

ISSN 2719-2415 (Online)

Case k =3: v = 2 as defined in (11), and (10) becomes

$$y(x) = \propto_0 (x) y_n + \propto_1 (x) y_{n+1} + \propto_2 (x) y_{n+2} + \propto_3 (x) y_{n+3} + h_n \beta_2(x) f_{n+2} + h_n^2 \lambda_3(x) g_{n+3} \tag{17}$$

Similarly, using the same procedure as in case k=2 in (12), the first derivative of the continuous interpolant is evaluated at η = {0, h_n, 3h_n} yields the following three discrete schemes:

$$y_{n+1} = -\frac{2}{65} h_n^2 g_{n+3} - \frac{56}{65} h_n f_{n+2} - \frac{34}{65} h_n f_{n+1} + \frac{17}{65} y_{n+3} + \frac{4}{5} y_{n+2} - \frac{4}{65} y_n$$

$$y_{n+2} = \frac{1}{9} h_n^2 g_{n+3} + \frac{13}{6} h_n f_{n+2} - \frac{17}{54} h_n f_n - \frac{68}{81} y_{n+3} + \frac{8}{3} y_{n+1} - \frac{67}{81} y_n \tag{18}$$

$$y_{n+3} = -\frac{18}{187} h_n^2 g_{n+3} + \frac{6}{11} h_n f_{n+3} + \frac{108}{187} h_n f_{n+2} + \frac{162}{187} y_{n+2} + \frac{27}{187} y_{n+1} - \frac{2}{187} y_n$$

4. CONVERGENCE AND STABILITY ANALYSIS

In this section, the convergence and stability analysis of the block second derivative modified generalized backward differentiation formulae blended with backward differentiation formulae derived in section two is carried out with results presented in tabular and graphical forms. These include: order, error constant, and zero stability, consistency and the regions of absolute stability.

4.1 Order of Second Derivative Methods for k= 2 and 3

The order and error constant of the discrete schemes in (16) and (18) are obtained or carried out in block form. Following Lambert (1973) and Fatunla (1980) the local truncation error associated with the k-step second derivative multistep method (2) is the linear difference operator L defined as

$$L[y(x):h] = \sum_{i=0}^{k=0} \{\alpha_j y(x + jh_n) - h_n \beta_j y'(x + jh_n) - h_n^2 \beta_j y'(x + jh_n)\} \tag{19}$$

where y(x) is an arbitrary function, continuously differentiable on [a, b]. Expanding $y(x + jh_n)$ and its derivative $y'(x + jh_n)$ as Taylor series about x, and collecting terms in (19) give

$$L[y(x):h_n] = C_0 y(x) + C_1 h_n y^1(x) + ... + C_q h_n y^q(x) + ... \tag{20}$$

ISSN 2719-2415 (Online)

where the constant C_q, q=0, 1, … are given as

$$
\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j \\
C_1 &= \sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j \\
C_2 &= \frac{1}{2}\sum_{j=0}^{k} j\alpha_j - \sum_{j=0}^{k} j\beta_j - \sum_{j=0}^{k} \lambda_j \\
&\vdots \\
C_q &= \frac{1}{q}\sum_{j=0}^{k} j^q\alpha_j - \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{q-1}\beta_j - \frac{1}{(q-2)!}\sum_{j=0}^{k} j^{q-2}\lambda_j, q = 3, 4, \ldots \\
&\vdots
\end{aligned}
\tag{21}
$$

The method in (16) expressed in the form of (19) produces the values of the continuous coefficients of the method as:

$$
\begin{aligned}
&\alpha_0 = \left(\frac{1}{16}, \frac{1}{17}\right)\alpha_1 = \left(1, -\frac{16}{17}\right), \alpha_2 = \left(-\frac{17}{16}, 1\right), \beta_0 = (0, 0) \\
&\beta_1 = \left(-\frac{1}{2}, \frac{8}{17}\right), \beta_2 = \left(-\frac{5}{8}, \frac{10}{17}\right), \lambda_2 = \left(\frac{1}{8}, -\frac{2}{17}\right)
\end{aligned}
\tag{22}
$$

Substituting the values of the continuous coefficients in (22) into (21) and solving gives $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = 0$ but $C_6 \neq 0$ that is $C_{p+2} \neq 0$

Thus the methods (16) are of uniform order p=4 with error constant $C_6 = \left(\frac{-1}{240}, \frac{1}{255}\right)$.

The method is also consistent since p = 4.

Using the same approach as in k=2, the order and error constants of the SDMGBDF blended with BDF for k = 3 is obtained and displayed in Table 1.

ISSN 2719-2415 (Online)

Table 1: Order and Error Constant of Second Derivative Methods for k = 2 and 3

Step No.	Evaluation points	Order	Error Constant
k=2	$y'(x_n), y'(x_{n+1})$	4	$-4.2\times10^{-3}, 3.9\times10^{-3}$
k=3	$y'(x_n), y'(x_{n+1})$ $y'(x_{n+3})$	5	$3.4\times10^{-3}, -1.8\times10^{-2}$, 1.6×10^{-3}

4.2 Zero Stability of Second Derivative Methods for k = 2 and 3

To determine the zero stability of the methods in (16), we express our methods in the form of Lambert (1973) to give:

$$y_{n+1} = \frac{1}{8}h_n^2 g_{n+2} - \frac{5}{8}h_n f_{n+2} - \frac{1}{2}h_n f_{n+1} + \frac{17}{16}y_{n+2} + \frac{1}{16}y_n$$

$$\rho(r) = r - \frac{17}{16}r^2 = r(1 - \frac{17}{16}r)$$

$$r = 0, 1 - \frac{17}{16}r = 0, r = \frac{16}{17}$$

$$y_{n+2} = -\frac{2}{17}h_n^2 g_{n+2} + \frac{10}{17}h_n f_{n+2} + \frac{8}{17}h_n f_{n+1} + \frac{16}{17}y_{n+1} + \frac{1}{17}y_n$$

$$\rho(r) = r^2 - \frac{16}{17}r = r(r - \frac{16}{17})$$

$$r = 0, r = \frac{16}{17}$$

Since r=0, $r = 0, r = \frac{16}{17}$ by Lambert (1963) the methods in (16) are zero stable.

Using the same approach for when k=2 of the SDMGBDF with BDF for k=3, the methods are all consistent and zero stable. Hence by Henrici (1962), the SDMGBDF blended with BDF are all convergent.

ISSN 2719-2415 (Online)

4.3 Regions of Absolute Stability of Second Derivative Methods

In this section, the regions of absolute stability of the schemes derived in section two in block form are plotted using the approach of Okuonghae & Ikhile (2011) by expressing the methods derived in section two into general linear methods in the form of:

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}\begin{pmatrix} y_{n+1} \\ \vdots \\ y_{n+k} \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \cdots & \vdots \\ b_{k1} & \cdots & b_{kk} \end{pmatrix}\begin{pmatrix} y_{n-1} \\ \vdots \\ y_{n} \end{pmatrix} + \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \cdots & \vdots \\ c_{k1} & \cdots & c_{kk} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ \vdots \\ f_{n+k} \end{pmatrix} +$$

$$\begin{pmatrix} d_{11} & \cdots & d_{1k} \\ \vdots & \cdots & \vdots \\ d_{k1} & \cdots & d_{kk} \end{pmatrix}\begin{pmatrix} f_{n-1} \\ \vdots \\ f_{n} \end{pmatrix} + \begin{pmatrix} w_{11} & \cdots & w_{1k} \\ \vdots & \cdots & \vdots \\ w_{k1} & \cdots & w_{kk} \end{pmatrix}\begin{pmatrix} g_{n+1} \\ \vdots \\ g_{n+k} \end{pmatrix} + \begin{pmatrix} u_{11} & \cdots & u_{1k} \\ \vdots & \cdots & \vdots \\ u_{k1} & \cdots & u_{kk} \end{pmatrix}\begin{pmatrix} g_{n-1} \\ \vdots \\ g_{n} \end{pmatrix} \tag{23}$$

where,

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \cdots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}, B = \begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \cdots & \vdots \\ b_{k1} & \cdots & b_{kk} \end{pmatrix}, C = \begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \cdots & \vdots \\ c_{k1} & \cdots & c_{kk} \end{pmatrix},$$

$$D = \begin{pmatrix} d_{11} & \cdots & d_{1k} \\ \vdots & \cdots & \vdots \\ d_{k1} & \cdots & d_{kk} \end{pmatrix}, W = \begin{pmatrix} w_{11} & \cdots & w_{1k} \\ \vdots & \cdots & \vdots \\ w_{k1} & \cdots & w_{kk} \end{pmatrix}, U = \begin{pmatrix} u_{11} & \cdots & u_{1k} \\ \vdots & \cdots & \vdots \\ u_{k1} & \cdots & u_{kk} \end{pmatrix}$$

The elements of A, B, C, D, W, and U are substituted into the characteristic equation (24).

$$\det(r(A - Cz - Wz^2) - B - Dz - Uz^2)) = 0 \tag{24}$$

Solving (24) in a maple environment yields the stability polynomial of the method and using a MATLAB code, the regions of absolute stability of the Second Derivative Linear Methods are plotted

4.3.1 Region of Absolute Stability of Second Derivative for k = 2 and 3

Using the approach in Okuonghae & Ikhile (2011), the methods in (16) are expressed in the form of (23) gives:

$$\begin{pmatrix} 1 & -\frac{17}{16} \\ -\frac{16}{11} & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{16} \\ 0 & -\frac{5}{11} \end{pmatrix}\begin{pmatrix} y_{n-1} \\ y_{n} \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} & -\frac{5}{8} \\ 0 & \frac{8}{11} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2}{11} \end{pmatrix}\begin{pmatrix} f_{n-1} \\ f_{n} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{8} \\ 0 & -\frac{2}{11} \end{pmatrix}\begin{pmatrix} g_{n+1} \\ g_{n+2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} g_{n-1} \\ g_{n} \end{pmatrix} \tag{25}$$

where,

$$A=\begin{pmatrix}1 & -\frac{17}{16}\\ -\frac{16}{11} & 1\end{pmatrix}, B=\begin{pmatrix}0 & -\frac{1}{16}\\ 0 & -\frac{5}{11}\end{pmatrix}, C=\begin{pmatrix}-\frac{1}{2} & -\frac{5}{8}\\ 0 & \frac{8}{11}\end{pmatrix}$$

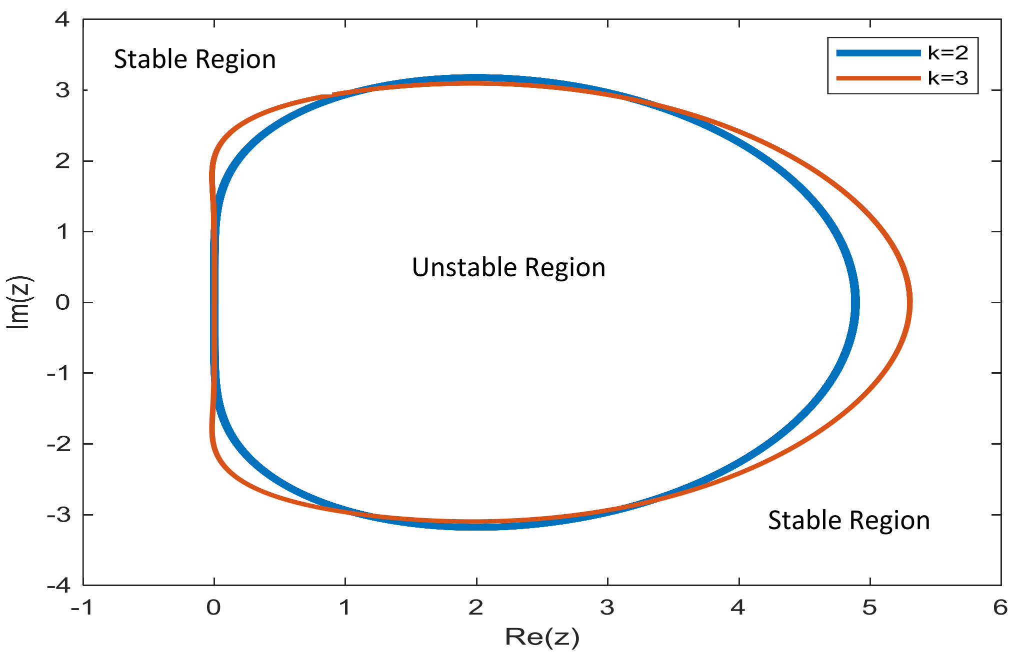
$$D=\begin{pmatrix}0 & 0\\ 0 & -\frac{2}{11}\end{pmatrix}, W=\begin{pmatrix}0 & \frac{1}{8}\\ 0 & -\frac{2}{11}\end{pmatrix}, U=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

Substituting the values of the matrices A, B, C, D, W, and U into (20) in a MATLAB environment produces the stability polynomial of the SDMGBDF blended with BDF for k = 2 in (26):

$$\rho(r)=-\frac{6}{11}r^2+\frac{15}{22}r^2z-\frac{4}{11}r^2z^2+\frac{6}{11}r+\frac{9}{22}rz+\frac{1}{11}r^2z^3+\frac{1}{11}rz^2 \qquad (26)$$

The stability polynomial in (26) plotted in a MATLAB environment produces the region of absolute stability of the SDMGBDF blended with BDF for k=2 and 3 as shown in figure 3.

Figure 1: Regions of Absolute Stability of second derivative Methods for k=2 and 3



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5. MODEL SIMULATION

In this section, three existing real-life problems with chaotic behaviours are solved numerically using our new methods and results compared with those obtained from existing methods in literature and the MATLAB ODEs solver. We considered the following problems in Wang et al., (2020), Ahmed et al., (2022) and Huang & Li (2018).

5.1 Numerical Simulation of Models

5.1.1 Model 1: Stock Price, Interest Rate, and Investor Sentiment

This system captures a chaotic system which described the wealth and inflation with heterogeneous agents. The system is modeled by the following set of IVPs.

$$
\begin{aligned}
y_1' &= \alpha y_1 + \beta y_2 - y_1 y_2 y_3 + \delta(y_1^2 + y_2^2 + y_3^2) \\
y_2' &= \zeta y_2 + \eta y_3 + \theta y_1 y_2 - \iota(y_1^2 + y_2^2 + y_3^2) \\
y_3' &= \lambda y_3 + \mu y_1 - \nu y_2 + \xi(y_1^2 + y_2^2 + y_3^2)
\end{aligned}
$$

Initial conditions: $y_1(0) = 10$, $y_2(0) = 10$, $y_1(0) = 0.5$, $x\epsilon[0,20]$. This system was solved for the parameters: α=0.1, β=0.2, γ=0.3, δ=0.01, ζ=0.05, η=0.1, θ=0.2, ι=0.01, λ=0.1, μ=0.2, ν=0.3, ξ=0.01, Source: Wang et al., (2020).

5.1.2 Model 2: Wealth, Inflation, and Stock Price

This system captures a chaotic system which described the wealth and inflation with heterogeneous agents. The system is modeled by the following set of IVPs.

$$
\begin{aligned}
y_1' &= \varphi y_1 + \psi y_2 - y_1 y_2 \\
y_2' &= \tau y_2 + \upsilon \varsigma - \theta y_1 \\
y_3' &= \lambda y_3 + \mu y_1 - \phi y_2 + \delta(y_1^2 + y_2^2 + y_3^2)
\end{aligned}
$$

Initial conditions: $y_1(0) = 100$, $y_2(0) = 10$, $y_1(0) = 0.5$, $x\epsilon[0, 20]$. This system was solved for the parameters: φ=0.05, ψ=0.1, ρ=0.02, τ=0.01, υ=0.1, θ=0.01, λ=0.1, μ=0.2, ϕ=0.3, δ=0.01.Source: Ahmed et al., (2022).

5.1.3 Model 3: Interest Rate, Investment Cost, and Price Exponent

This model descried the chaotic financial system and is given by

$$
\begin{aligned}
y_1' &= y_3 + y_2 y_1 - a y_1 \\
y_2' &= 1 - b y_2 - y_1^2 \\
y_3' &= -y_1 - c y_3
\end{aligned}
\qquad \text{T}
$$

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Initial conditions: $y_1(0) = 1$, $y_2(0) = 2$, $y_1(0) = 0.9$, $x\epsilon[0, 200]$. This system was solved for the parameters, a=0.95, b=0.2, c=1.5, Source: Huang & Li (2018).

5.2 Solutions to Numerical Simulation of Models

Tables 2-4 shows the values of each component of models solved at selected times. These values were obtained using our newly developed scheme and ODE23s. The results obtained using all the methods are in good agreement for all the selected times presented on the tables and the solution curves. Our CPU time is smaller compared with ODE23s solver.

Table 2: Numerical Simulation of model 1 using SDMGBF Blended with BDF k =2, 3 and ODE 23s

Methods	SDMGBF Blended with BDF k = 2	SDMGBF Blended with BDF k = 3	ODE 23s
CPU TIME	0.146s	0.186s	0.926s
Step Size	h=0.1-0.001	h=0.1-0.001	h=0.001
y1(t)			
2	4.707971	4.707971	4.707971
4	5.440316	5.440316	5.440316
6	10.551679	10.551679	10.551679
8	8.014881	8.014881	8.014881
10	6.541885	6.541885	6.541885
12	6.836965	6.836965	6.836965
14	7.6555241	7.6555241	7.6555241
16	8.180731	8.180731	8.180731
18	8.230986	8.230986	8.230986
20	8.046312	8.046312	8.046312
y2(t)			
2	10.386076	10.386076	10.386076

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4	10.930979	10.930979	10.930979
6	11.318991	11.318991	11.318991
8	11.647313	11.647313	11.647313
10	11.924414	11.924414	11.924414
12	12.134560	12.134560	12.134560
14	12.280555	12.280555	12.280555
16	12.380495	12.380495	12.380495
18	12.449986	12.449986	12.449986
20	12.498972	12.498972	12.498972
y2(t)			
2	-0.186498	-0.186498	-0.186498
4	0.258307	0.258307	0.258307
6	0.090681	0.090681	0.090681
8	-0.070051	-0.070051	-0.070051
10	0.011276	0.011276	0.011276
12	0.086964	0.086964	0.086964
14	0.102674	0.102674	0.102674
16	0.080648	0.080648	0.080648
18	0.055638	0.055638	0.055638
20	0.043734	0.043734	0.043734

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Figure 2: Phase Space Plots of Model 1 using SDMGBDF blended with BDF k=2 Red and ode23s Black

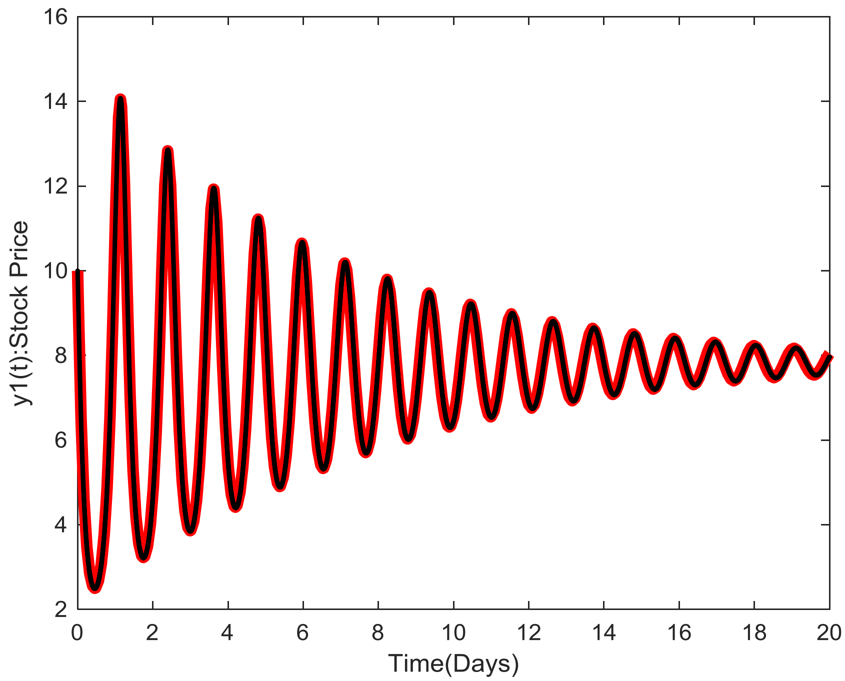

Figure 3: Phase Space Plots of model 1 using SDMGBDF blended with BDF k=2 Red and ode23s Black.

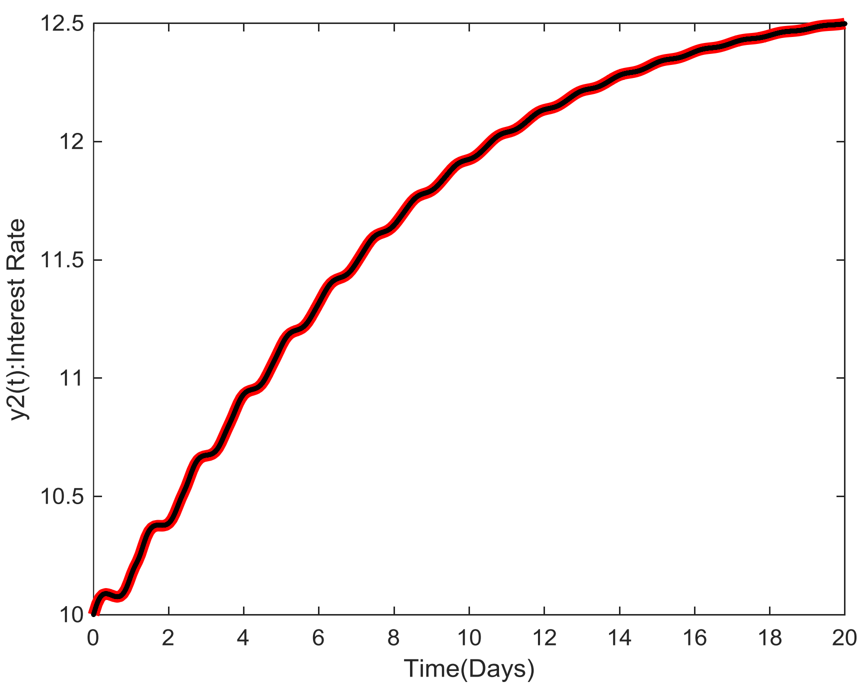

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Figure 4: Phase Space Plots of model 1 using SDMGBDF blended with BDF k=2 Red and ode 23s Black.

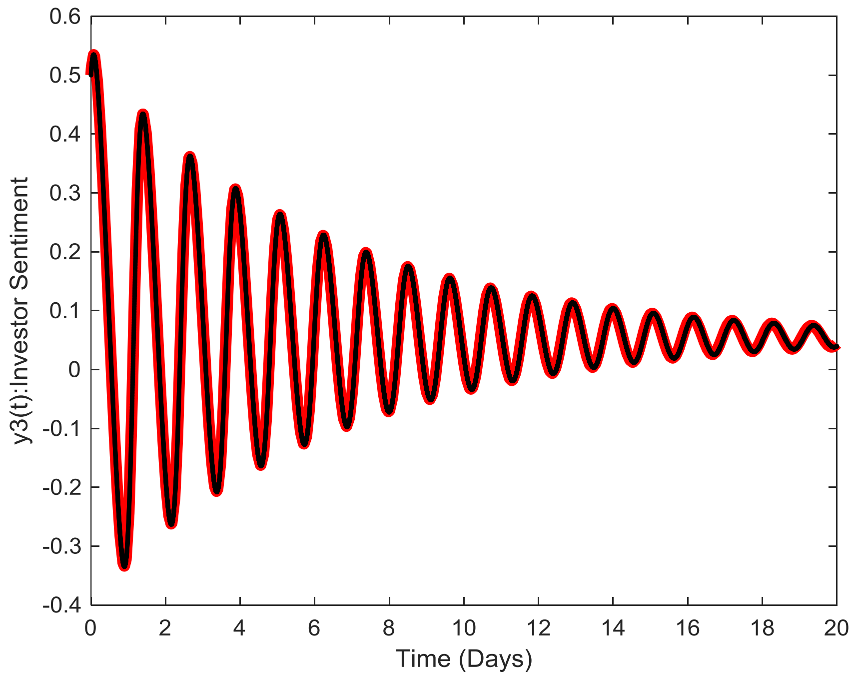

Figure 5: Phase Portraits of model 1 using SDMGBDF blended with BDF k=2 Blue and ode 23s Red.

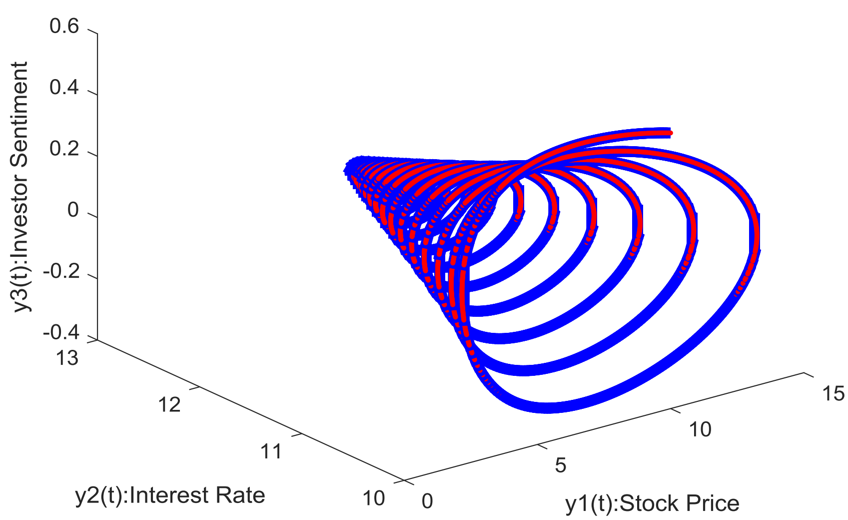

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Figure 6: Phase Portraits of model 1 using SDMGBDF blended with BDF k=2 Blue and ode 23s Red

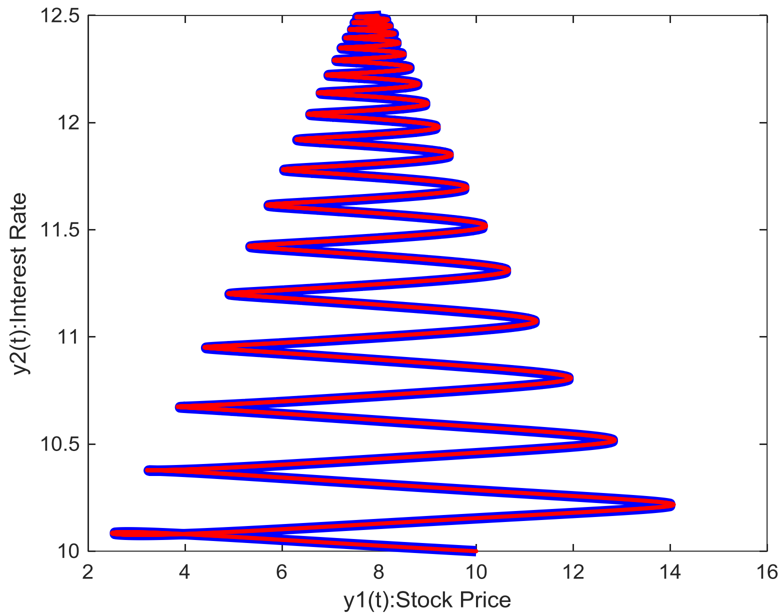

Figure 7: Phase Portraits of model 1 using SDMGBDF blended with BDF k=2 Blue and ode 23s Red.

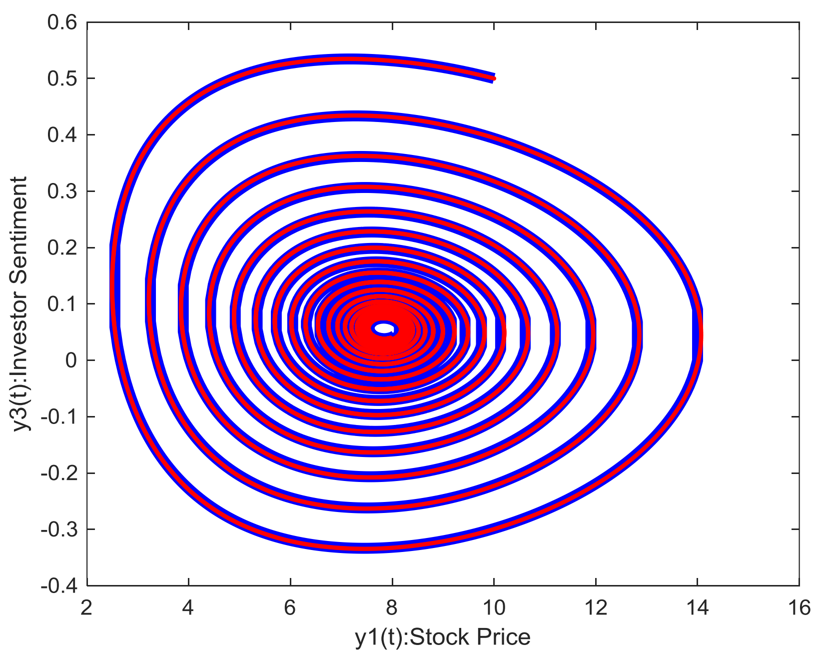

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Figure 8: Phase Portraits of model 1 using SDMGBDF blended with BDF k=2 Blue and ode 23s Red

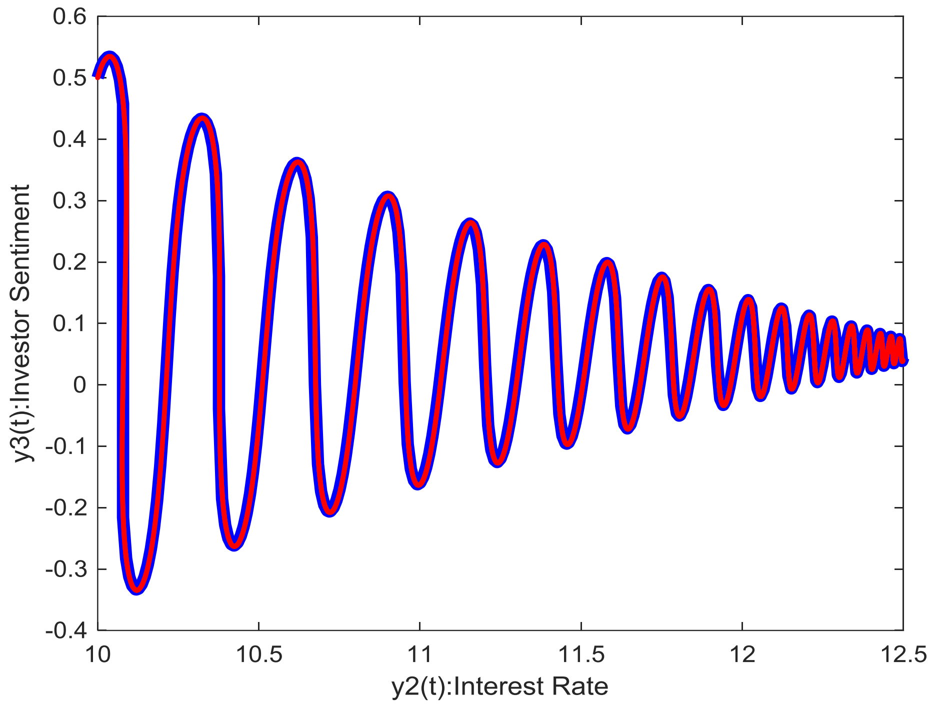

Table 3: Numerical Solutions of model 2 using SDMGBF Blended with BDF k =2, 3 and ODE23s

Methods	SDMGBF Blended with BDF k = 2	SDMGBF Blended with BDF k = 3	ODE 23s
CPU TIME	0.153s	0.265s	0.314s
Step Size	h=0.1-0.01	h=0.1-0.01	h =0.001
y1(t)			
2	0.0008404	0.0008404	0.0008404
4	0.0009144	0.0009144	0.0009144
6	0.0015623	0.0015623	0.0015623
8	0.0013021	0.0013021	0.0013021

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10	0.0013673	0.0013673	0.0013673
y2(t)			
2	-0.0000500	-0.0000500	-0.0000500
4	-0.0008044	-0.0008044	-0.0008044
6	0.00039478	0.00039478	0.00039478
8	0.00011663	0.00011663	0.00011663
10	0.00012215	0.00012215	0.00012215
y3(t)			
2	-0.0013923	-0.0013923	-0.0013923
4	-0.0012813	-0.0012813	-0.0012813
6	0.0011989	0.0011989	0.0011989
8	0.0054729	0.0054729	0.0054729
10	0.0001540	0.0001540	0.0001540

Figure 9: Phase Space Plots of Model 2 using SDMGBDF blended with BDF k=3 Purple and ode 23s Black

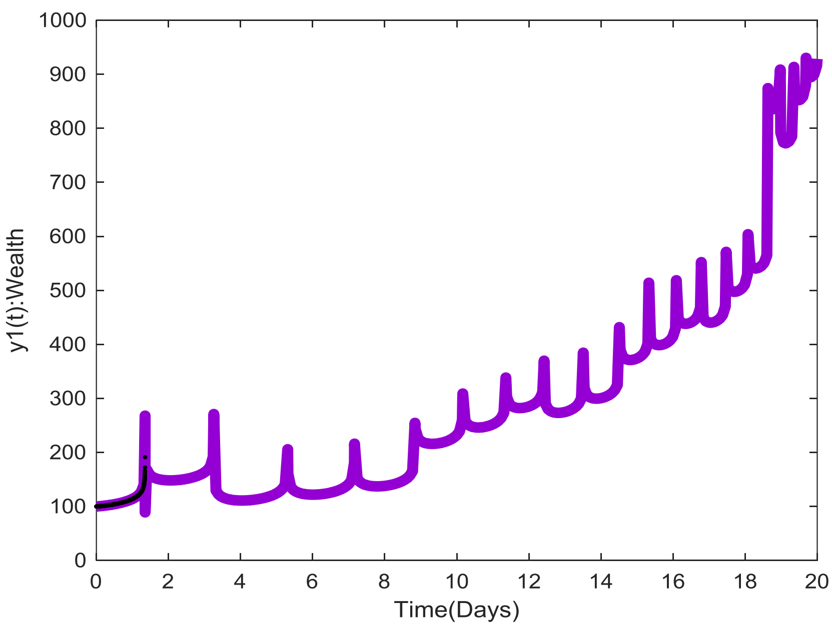

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Figure 10: Phase Space Plots of Model 2 using SDMGBDF blended with BDF k=3 Purple and ode 23s Black.

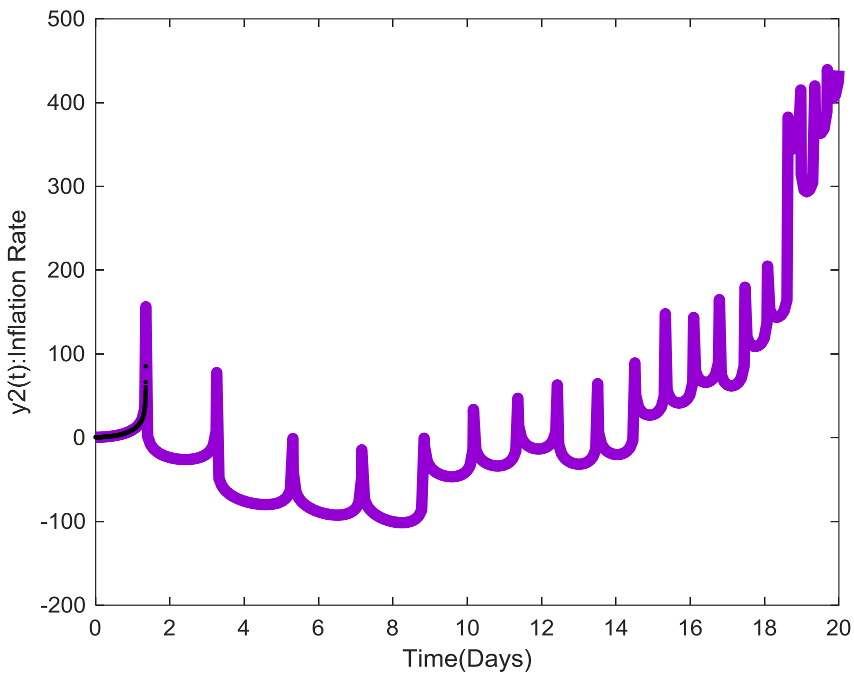

Figure 11: Phase Space Plots of Model 2 using SDMGBDF blended with BDF k=3 Purple and ode 23s Black.

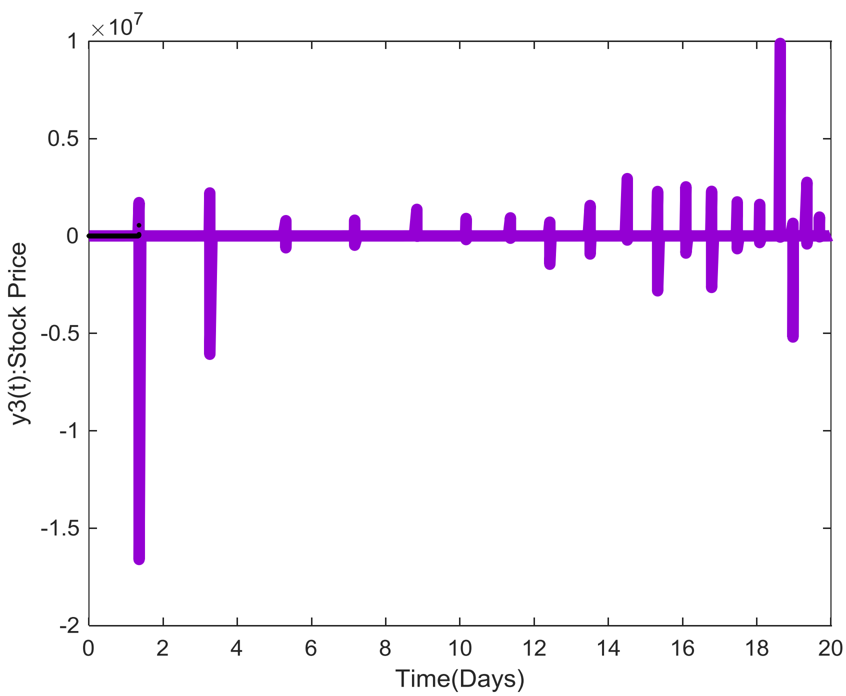

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Figure 12: Phase Portrait Plots of Model 2 using SDMGBDF blended with BDF k=3 Purple and ode 23s Black

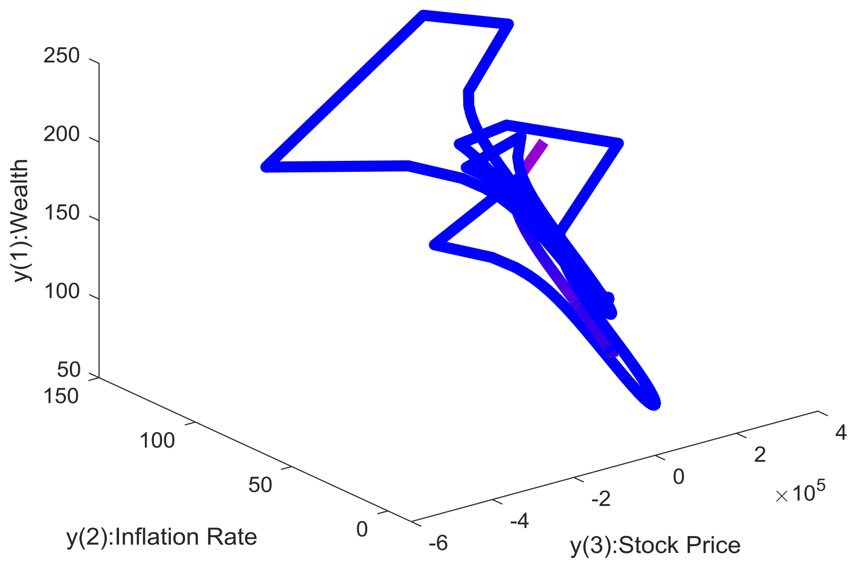

Figure 13: Phase Portraits of Model 2 using SDMGBDF blended with BDF k=3 Blue and ode23s Purple

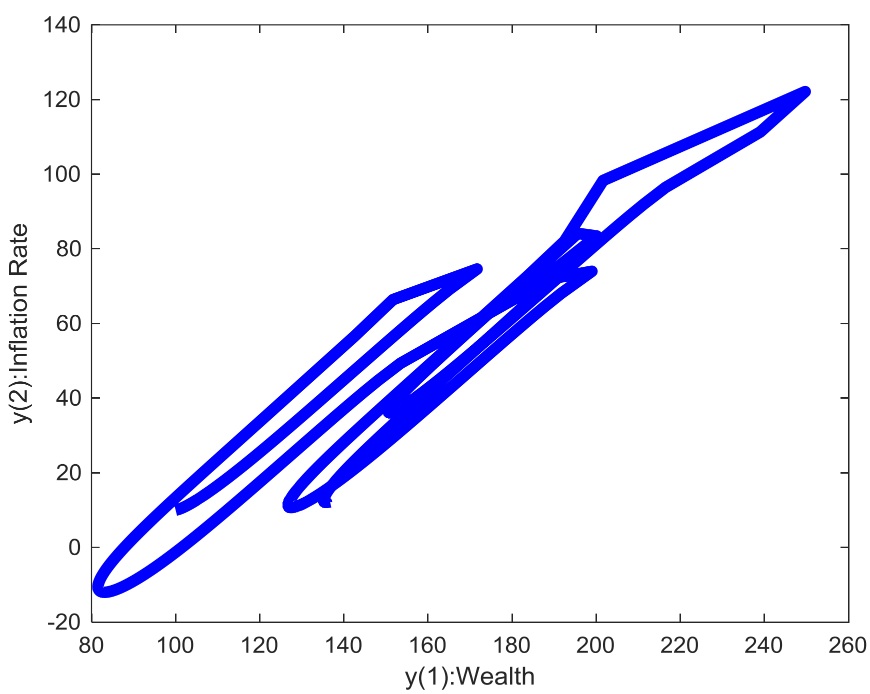

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Figure 14: Phase Portraits of Model 2 using SDMGBDF blended with BDF k=3 Blue and ode23s Purple

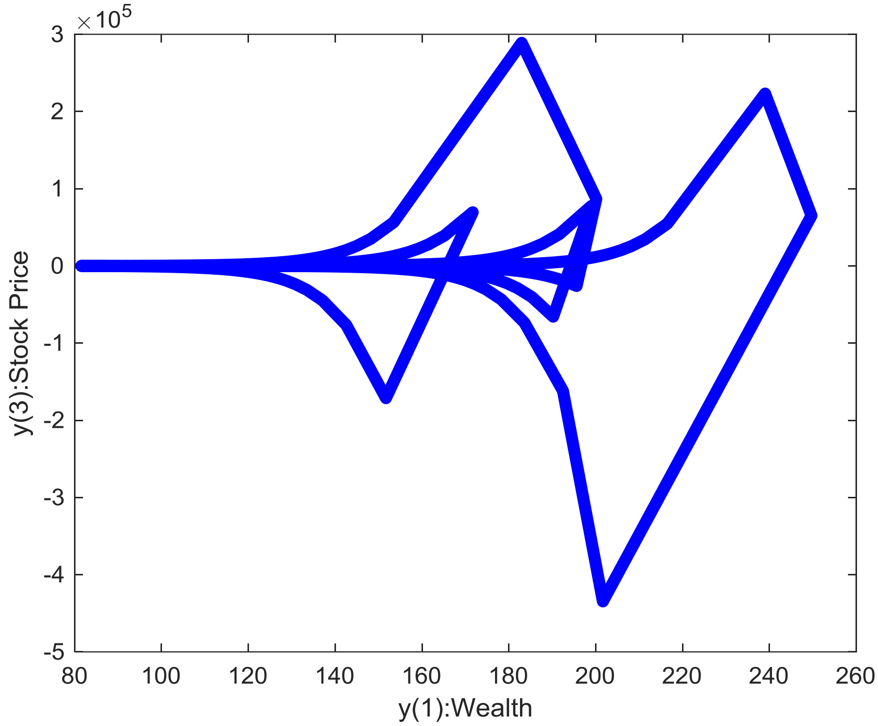

Figure 15: Phase Portraits of Model 2 using SDMGBDF blended with BDF k= 3 Blue and ode23s Purple

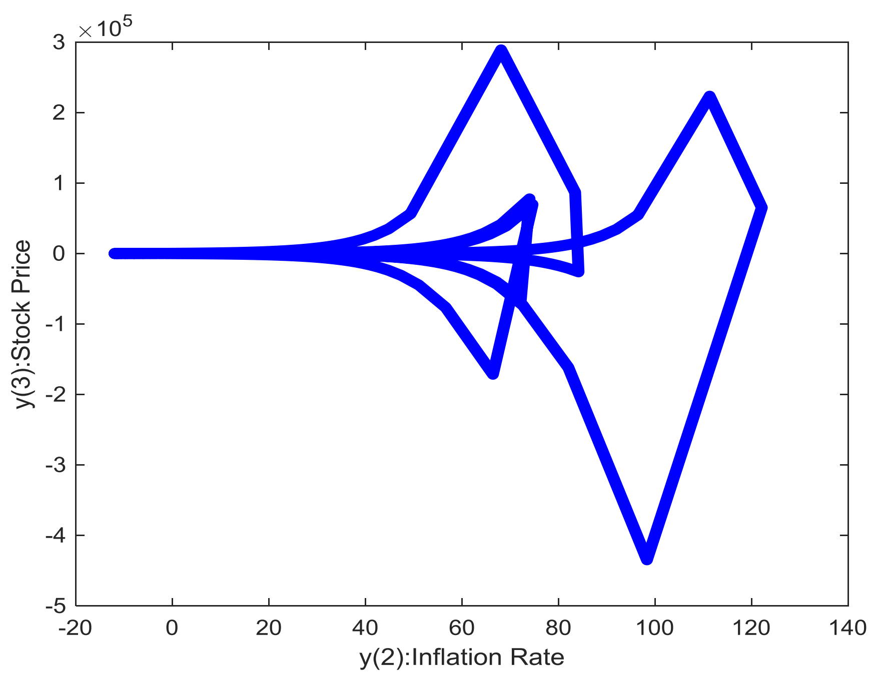

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Table 4: Numerical Solutions of Model 3 using SDMGBF Blended with BDF k =2, 3 and ODE 23s

Methods	SDMGBF Blended with BDF k = 2	SDMGBF Blended with BDF k = 3	ODE 23s
CPU TIME	0.125s	0.144s	0.142s
Step Size	h=0.1-0.01	h=0.1-0.01	h=0.1-0.01
y1(t)			
20	-0.011461	-0.011461	-0.011461
40	-1.501012	-1.501012	-1.501012
60	-0.019546	-0.019546	-0.019546
80	1.626262	1.626262	1.626262
100	0.086041	0.086041	0.086041
y2(t)			
20	1.339326	1.339326	1.339326
40	2.212641	2.212641	2.212641
60	1.762742	1.762742	1.762742
80	1.372151	1.372151	1.372151
100	1.035769	1.035769	1.035769
y3(t)			
20	0.211258	0.211258	0.211258
40	0.547505	0.547505	0.547505
60	-0.112985	-0.112985	-0.112985
80	-0.768645	-0.768645	-0.768645
100	0.1772405	0.1772405	0.1772405

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Figure 16: Phase Space Plots of Model 3 using SDMGBDF blended with BDF k=3 Purple and ode 23s Red.

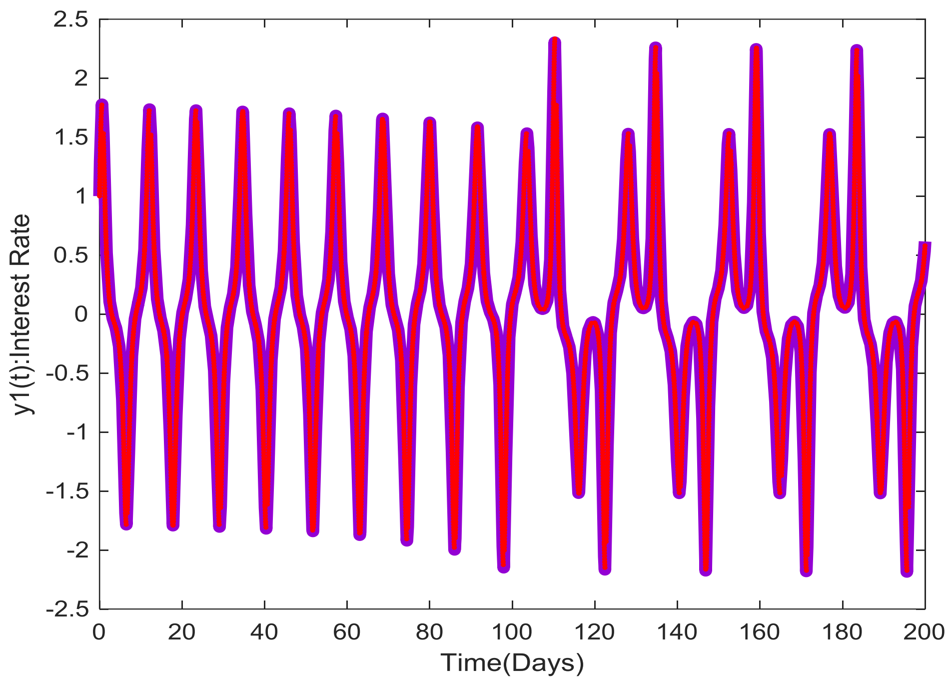

Figure 17: Phase Space Plots of Model 3 using SDMGBDF blended with BDF k=3 Purple and ode 23s Red.

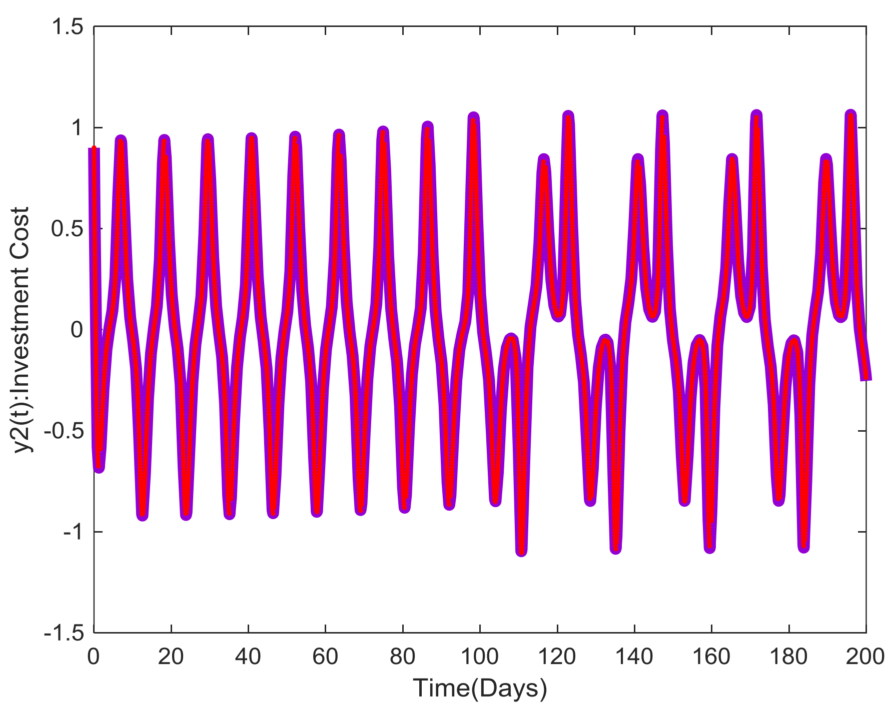

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Figure 18: Phase Space Plots of Model 3 using SDMGBDF blended with BDF k=3 Purple and ode 23s Red.

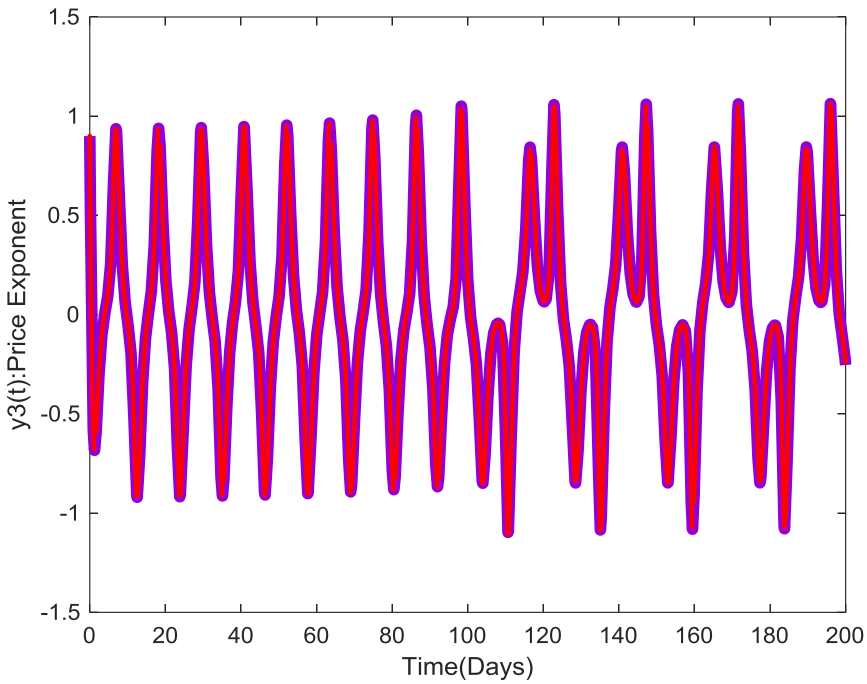

Figure 19: Phase Portrait Plots of Model 3 using SDMGBDF blended with BDF k=3 Purple and ode 23s Red.

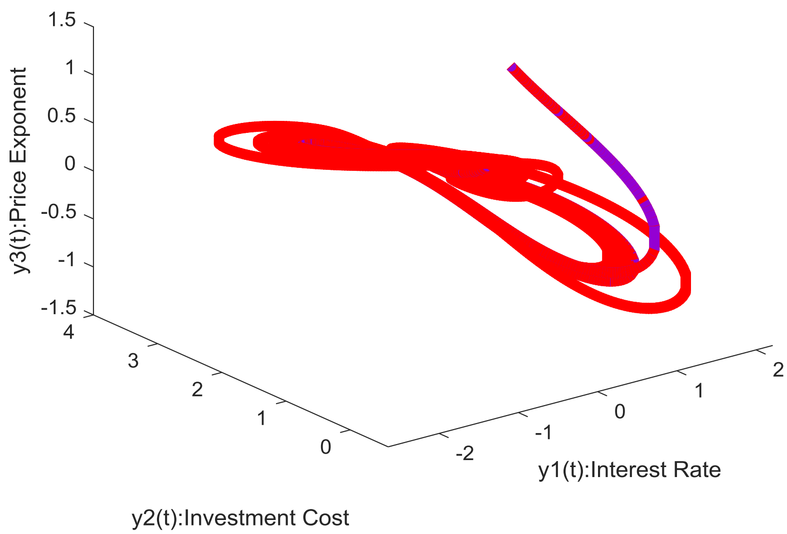

Figure 20: Phase Portraits of Model 3 using SDMGBDF blended with BDF k=3 Purple and ode23s Red.

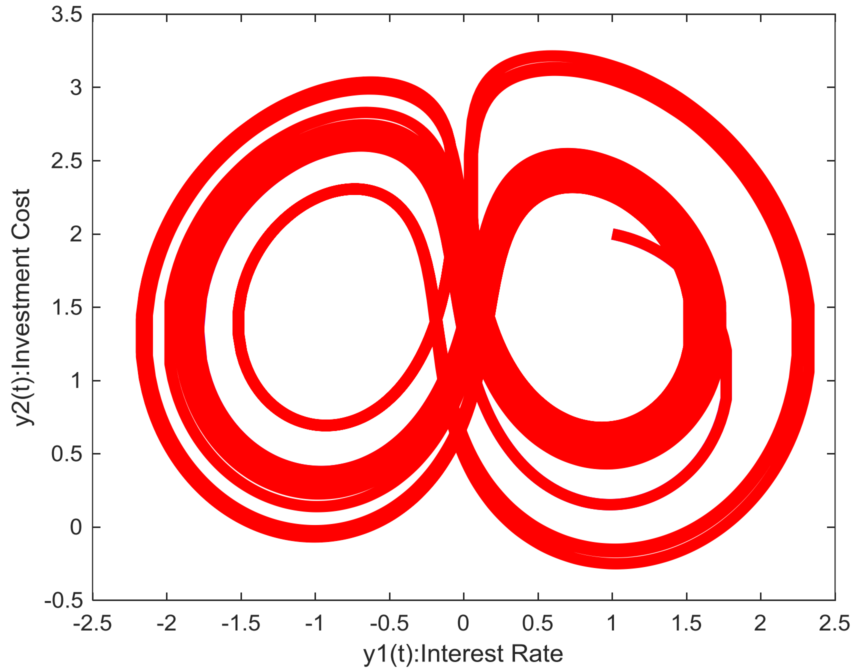

Figure 21: Phase Portraits of Model 3 using SDMGBDF blended with BDF k=3 Purple and ode23s Red.

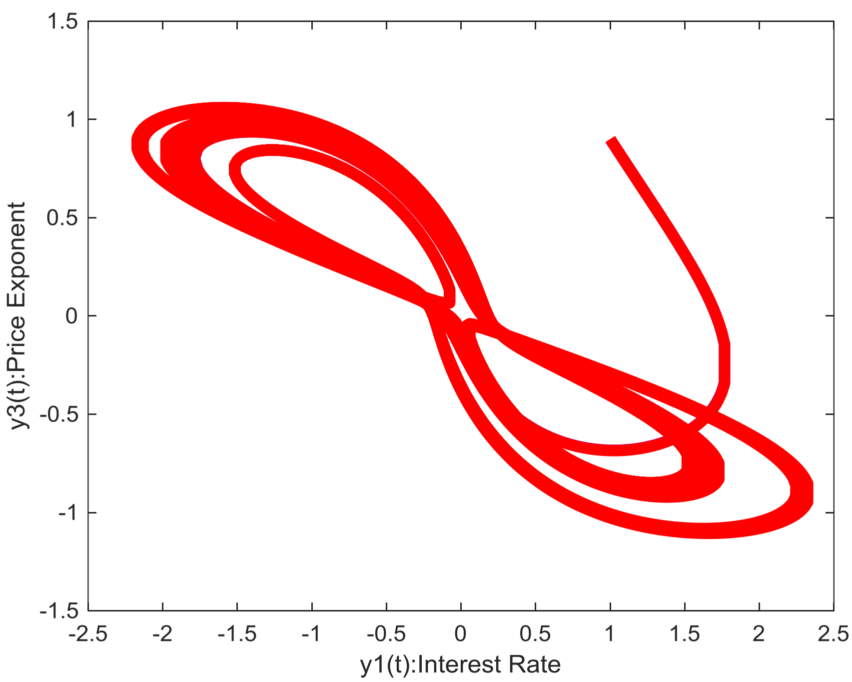

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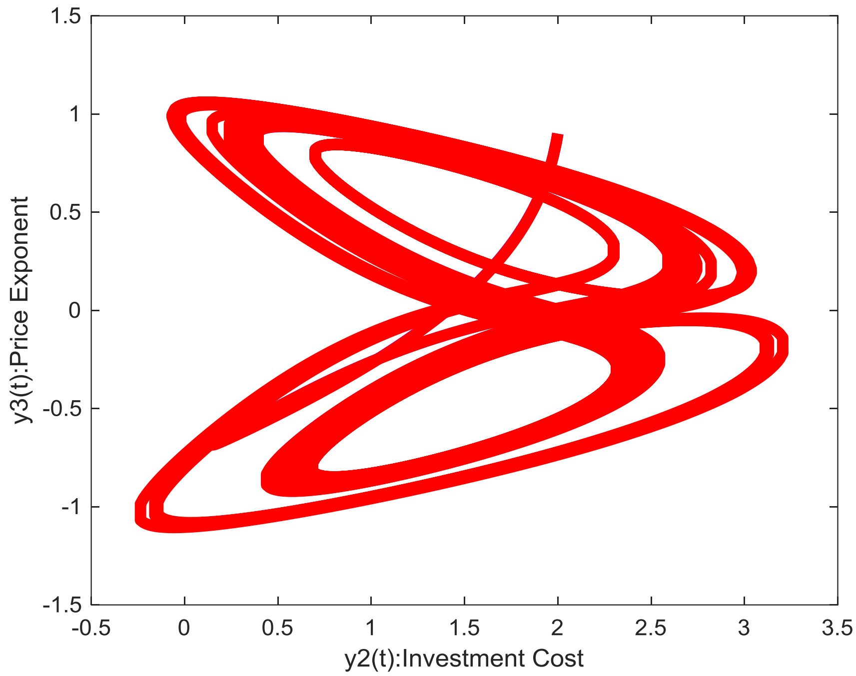
Figure 22: Phase Portraits of Model 3 using SDMGBDF blended with BDF k=3 Purple and ode23s Red.



6. DISCUSSION

In this section, we present the results of applying our newly developed methods and ode23s to financial chaotic models. The results show that the new methods accurately approximate the solutions of financial chaotic initial value problems (IVPs) of ordinary differential equations (ODEs). The phase spaces and phase portraits illustrate the chaotic behaviour of the various market regimes, including: Wealth, inflation rates, and stock prices (Model 1),

Stock prices, interest rates, and investor sentiment (Model 2), saving amount, per investment cost, and demand elasticity (Model 3). The chaotic behaviour in these markets regimes can lead to severe economic and financial consequences, including: Market crashes, Hyperinflation, Wealth redistribution, Economic instability, Unpredictable dynamics in wealth and inflation, increased inequalities, Systemic risk, and Global contagion (Stock & Watson, 2020). To mitigate these effects, effective tools include: Monetary policy adjustments, Scenario planning, Fiscal policy interventions, Diversification, Risk management strategies (Krugman, 2020).

The results also highlight the impact of various parameters on the systems behaviour, including: Saving amount (a), per investment cost (b), and Demand elasticity (c). Changes in these parameters can lead to increased chaotic behaviour, mirroring economic recessions.

ISSN 2719-2415 (Online)

7. CONCLUSION

This study aimed to develop and apply a convergent and A-stable method with variable step size for solving chaotic financial systems. The research questions, objectives, and statement of the problem guided the investigation.

The findings of the study provide affirmative answers to the research questions;

Can the conventional second derivative LMM with variable step size solve nonlinear IVPs of ODEs with chaotic properties? Yes, the developed method demonstrated convergence and A-stability. Are the regions of absolute stability of the conventional second derivative LMMs stable? Yes, the method showed stability in solving chaotic financial systems.

Is numerical procedure involved in computing solutions tedious and time-consuming when using variable step sizes? No, the comparison with ode23s showed that the developed method required smaller CPU time indicating that it is not tedious for the system. Can global convergence be achieved using longer intervals without splitting? Yes, the study demonstrated global convergence without splitting the interval of integration.

The study achieved its objective of deriving second derivative conventional forms of variable step size LMMs for solving chaotic IVPs of ODEs. The developed method effectively addressed the statement of the problem by providing a convergent and A-stable solution to chaotic financial systems. The findings of this study confirm that financial markets exhibit chaotic behaviour characterized by unpredictability, sensitivity to initial conditions, and complex dynamics. Specifically, the results show that: This means: Long-term predictions are impossible due to unpredictability of chaotic financial systems. Small changes in investor sentiment can drastically affect stock prices, highlighting the sensitivity of financial markets to initial conditions. Market volatility and wealth are interconnected and influence one another, leading to complex dynamics. Inflation rate fluctuations significantly impact overall market dynamics, leading to chaotic market regimes. The study's finding have significant implications for financial institutions, policymakers, and investors, emphasizing the need for adaptive strategies to note that numerical analysis should be complemented with empirical studies to ensure the accuracy and relevance of the findings.

ISSN 2719-2415 (Online)

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ISSN 2719-2415 (Online)