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Chromatic Polynomials of S -Valued Graphs

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ABSTRACT

In his monograph "Semirings and their application" Jonathan Golan has introduced the notion of R – valued graphs where R is a semiring. In the year 2015 Chandramouleeswaran and his scholars introduced the notion of semiring valued graphs (briefly called S-valued graphs). Many research has been carried out in S-valued graphs such as vertex domination, edge domination, vertex-edge domination, colouring of S-valued graphs and so on. This paper discusses the concept of vertex S-chromatic polynomial associated with a given S-valued graphs.

1. INTRODUCTION

Coloring of graphs is the most important concept in which we partition the vertex (edge) set of any associated graph, so that the adjacent vertices (edges) belong to different sets of the partitions. In other words, colouring problem is considered as grouping the items of interest as few groups as possible, so that incompatible items are in different groups. Colouring of graphs is one of the most important research area of combinatorial optimization due to its wide applications in real life, viz. Management sciences, wiring printed circuits, resource allocation, scheduling problems, etc. These problems are modelled by appropriate crisp graphs and solved by colouring of these graphs. In the conventional graph colouring problem, minimum number of colours is given to the edges of the graphs such that no two adjacent edges have the same colours.

A k-colouring of graphs G is an assignment of integers $1,2,\cdots,k$ (the colours) to the vertex of G in such a way that the neighbours receive different integers (colours). The chromatic number of G is the smallest positive integer k such that G has a k-colouring.

The theory of semirings was first introduced by Vandiver,(1934) while studying the theory of ideals in commutative rings. In his book titled "The theory of semirings and their applications" Jonathan Golan (1999) introduced the notion of R-valued graph on any non empty set V where R is an arbitrary semiring. The elements of V are called the vertices and a R-valued relation g on V is called an R-valued graph G. He has assigned the nonzero values from the semiring R to the edges of G. Thus, Golan assigned the values from a semiring R only to the edges of G. Motivated by this, in the year 2015, Chandramouleeswaran and others introduced the notion of Semiring-valued graphs (simply called S-valued graphs)(Rajkumar et.al, 2015). That is, graphs G whose vertices and edges are labelled with values from a semiring S, using the canonical pre-order existing in the semiring to compare the S-values of the end vertices of any edge, assigned the S-value for the edge under consideration.

In our earlier work, we studied the vertex and edge colouring of S-valued graphs. This work defines the study of chromatic polynomials of a S-valued graph by considering different possible vertex colouring from a given colour class of k-colours. The vertex *S*-chromatic polynomials of some simple types of graphs have been obtained.

2. PRELIMINARIES

In this section, we recall basic definitions from the theory of S-valued graphs.

Definition 2.1. (Rajkumar et.al., 2015) Let G = (V, E $\subset$ V $\times$ V) be a given graph with V, E $\neq \phi$. For any semiring (S, +, $\cdot$), a semiring-valued graph (or a S – valued graph) G^S, is defined to be the graph G^S = (V, E, σ, ψ) where $\sigma : V \rightarrow S$ and $\psi : E \rightarrow S$ is defined to be

$$\psi(x,y)=\begin{cases}\min\{\sigma(x),\sigma(y)\} & if\ \sigma(x)\preceq\sigma(y)\ or\ \sigma(y)\preceq\sigma(x)\\ 0 & otherwise\end{cases}$$

for every unodered pair (x,y) of E $\subset$ V $\times$ V.

We call σ, a S – vertex set and ψ, a S – edge set of G^S.

Any vertex $v \in V(G^S)$ will be represented by $v(\sigma(v))$, where $\sigma(v) \in S$.

Similarly, any edge $e_i^j=(v_i,v_j)\in E(G^S)$ will be denoted by $e_i^j\left(\psi\left(e_i^j\right)\right)$.

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Definition 2.2 (Rajkumar et.al., 2015) Consider the S – valued graph $G^S = (V,E,\sigma,\psi)$.

The order of G^S is defined as

$$p_S = \left(\sum_{v \in V} \sigma(v), |V| \right)$$

The size of G^S is defined as

$$q_S = \left(\sum_{(u,v) \in E} \psi(u,v,), |E| \right)$$

Definition 2.3 (Jeyalakshmi & Chandramouleeswaran, 2015) Consider the S – valued graph $G^S = (V, E, \sigma, \psi)$. The open neighbourhood of v_i in G^S is defined as

$$N_S(v_i) = \left\{ \left(v_j, \sigma(v_j) \right) \ | \left(v_i, v_j \right) \in E, \psi\left(v_i, v_j \right) \in S \right\}.$$

The closed neighbourhood of v_i in G^S is defined as

$$N_S[v_i] = N_S(v_i) \cup \left\{ \left(v_i, \sigma(v_i) \right) \right\}.$$

Definition 2.4. (Mangala Lavanya et.al., 2016) Consider the S – valued graph $G^S = (V, E, \sigma, \psi)$. The open neighbourhood of e_i^j, denoted by $N_S\left(e_i^j\right)$, is defined to be the set

$$N_S\left(e_i^j\right) = \left\{ \left(e_k^l, \psi\left(e_k^l\right) \right) / \, e_i^j \textit{ and } e_k^l \textit{ are adjacent} \right\}$$

The closed neighbourhood of e_i^j, denoted by $N_S\left[e_i^j\right]$, is defined to be the set

$$N_S[e_i^j] = N_S(e_i^j) \cup \{(e_i^j, \psi(e_i^j))\}$$

Definition 2.5 (Mangala Lavanya et.al., 2016) Let $G^S = (V, E, \sigma, \psi)$ be a given S – valued graph. Let $D \subseteq V$. Then the scalar cardinality of D, denoted by $|D|_S$, is defined by

$$|D|_S = \sum_{v \in D} \sigma(v).$$

Let $F \subseteq E$. Then the scalar cardinality of F, denoted by $|F|_S$, is defined by

$$|F|_S = \sum_{e_i^j \in F} \psi\left(e_i^j\right)$$

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Definition 2.6. (Jeyalakshmi and Chandramouleeswaran, 2015) Consider the S – valued graph $G^S = (V, E, \sigma, \psi)$, where $V = \{v_1, v_2, ..., v_n\}$. A S – path between v_i and v_k is defined as the sequence.

$$\left(v_i, \sigma(v_i)\right)\left(e_i^j, \psi\left(e_i^j\right)\right)\left(v_j, \sigma\left(v_j\right)\right)...\left(e_{k-1}^k, \psi\left(e_{k-1}^k\right)\right)\left(v_k, \sigma(v_k)\right).$$

Definition 2.7 (Shriprakash and Chandramouleeswaran, 2017) Consider the S – valued graph $G^S = \{V, E, \sigma, \psi\}$. Let $C = \{c_1, c_2...\}$ be a set of colours. A colouring of G^S is given by a function $f : V \times V \to S \times C$ such that for all $v \in V$, $f(v,v) = (\sigma(v), c(v)), c(v) \in C.$

Definition 2.8 (Shriprakash and Chandramouleeswaran, 2017) Consider the S – valued graph $G^S = (V, E, \sigma, \psi)$. A colouring f on G^S is said to be an equi-weight (or vertex regular) proper colouring, if for all $v \in V, \sigma(v) = a,$ for some $a \in S$ and $c(v) \in C$ is different for adjacent vertices.

Definition 2.10 (Shriprakash and Chandramouleeswaran, 2017) A colouring $f : V \times V \to S \times C$ of $G^S = (V, E, \sigma, \psi)$ is said to be a total proper colouring, if $\forall\, v \in V$ both $\sigma(v) \in S$ and $c(v) \in C$ are different for adjacent vertices. That is, if for every pair of adjacent vertices $v_i v_j$ $i \neq j, \sigma(v_i) \neq \sigma(v_j)$ and $c(v_i) \neq c(v_j)$.

Definition 2.11 (Shriprakash and Chandramouleeswaran, 2017) Let $G^S = (V, E, \sigma, \psi)$ be a S – valued graph. The vertex chromatic number of G^S, denoted by $\chi_S(G^S)$, is defined to be

$$\chi_S(G^S) = \left(\min_{v \in V}(\sigma(v)), \min\{|C|\}\right).$$

3. VERTEX CHROMATIC POLYNOMIAL ON S – VALUED GRAPH

This section introduces the notion of vertex S – chromatic polynomial for a given S – valued graph. We start with the following definition.

Definition 3.1. Let $(S, +, \cdot, \preceq)$ be a given semiring and let $G^S = (V, E, \sigma, \psi)$ be a S – valued graph. Let C be a colour class of k colours. The vertex S-Chromatic polynomial of the S – valued graph G^S is defined as a function M_{G^S} that describes the number of ways we can get a proper vertex colouring of G^S, using the k colours form C.

The function M_{G^S} is called the vertex S-chromatic function.

Example 3.2 Consider the semiring $S = (\{0, a, b, c, d, e\}, +, \cdot, \preceq)$ with the following Cayley Tables:

Table 1

+	0	a	b	c	d	e
0	0	a	b	c	d	e
a	a	a	b	c	d	e
b	b	b	b	d	d	e
c	c	c	d	c	d	e
d	d	d	d	d	d	e
e	e	e	e	e	e	e

Table 2

.	0	a	b	c	d	e
0	0	0	0	0	0	0
a	0	0	0	a	a	b
b	0	a	b	a	b	b
c	0	0	a	c	c	e
d	0	a	b	c	d	e
e	0	c	e	c	e	e

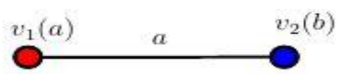
Table 3

$\preceq$	Elements of S
0	0,a,b,c,d,e
a	a,b,c,d,e
b	b,d,e
c	c,d,e
d	d,e
e	e

Let C = $\{c_1 = Red, c_2 = Blue, c_3 = Brown, ..., c_k\}$ be given colour class of k colours.

Let C = $\{c_1 = Red, c_2 = Blue, c_3 = Brown, ..., c_k\}$ be given colour class of k colours. Consider the semiring given above. Consider now the S – path P_2^S.

Figure 1



The vertex $v_1(a)$ can be coloured using any one of the k colours, say c_1, from the colour class C. After assigning the colour c_1 to the vertex $v_1(a)$, the vertex $v_2(b)$ can be coloured using any of the $(k-1)$ colours from C $\setminus \{c_1\}$. Thus the vertex S-chromatic polynomial of P_2^S is given by

$$M_{P_2^S}(k) = \left(\prod_{i=1}^{2} \sigma(v_i), k(k-1) \right)$$

Consider the S-path P_3^S.

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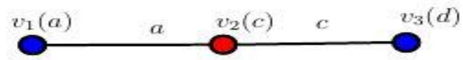
Figure 2

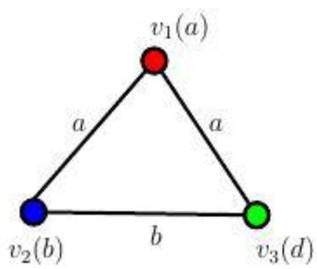


The middle vertex $v_2(c)$ can be coloured using any one of the k colours, say c_1 from the colour class C. After assigning the colour c_1 to the vertex $v_2(c)$, since the vertices $v_1(a)$ and $v_3(d)$ are not adjacent, they can be coloured using any of the ($k-1$) colours from C $\setminus\{c_1\}$. Thus the vertex S-chromatic polynomial of P_3^S is given by

$$M_{P_3^S}(k) = \left(\prod_{i=1}^{3}\sigma(v_i), k(k-1)^2\right)$$

Consider the S-cycle C_3^S.

Figure 3



The vertex $v_1(a)$ can be coloured using any one of the k colours, say c_1, from the colour class C. After assigning the colour c_1 to the vertex $v_1(a)$, the vertex $v_2(b)$ can be coloured using any of the ($k-1$) colours from C $\setminus\{c_1\}$. Since the vertex $v_3(d)$ is adjacent to both the vertices $v_1(a)$ and $v_2(b)$, it can be assigned any other colour, say c_3 from the colour class C $\setminus\{c_1, c_2\}$. Thus the vertex S-chromatic polynomial of C_3^S is given by

$$M_{C_3^S}(k) = \left(\prod_{i=1}^{3}\sigma(v_i), k(k-1)(k-2)\right).$$

From the above illustrations we obtain the results for any S-path and S-cycle as follows:

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Theorem 3.3

For any S-path P_n^S, the vertex S-chromatic polynomial is given by

$$M_{P_n^S}(k) = \left(\prod_{i=1}^{n} \sigma(v_i), k^{\left\lceil \frac{n}{2} \right\rceil} (k-1)^{\left\lceil \frac{n}{2} \right\rceil} \right)$$

Proof:

Consider the S-path P_n^S. First we colour the vertex $(v_2, \sigma(v_2))$ with any one of the k-colours from C. Without loss of generality, let us choose it to be c_1. Now the vertices $(v_1, \sigma(v_1))$ and $(v_3, \sigma(v_3))$ are not adjacent but both of them are adjacent to $(v_2, \sigma(v_2))$. Hence they can be assigned the same colour from any one of the $(k-1)$ colours from C $\setminus \{c_1\}$. The vertex $(v_4, \sigma(v_4))$ is adjacent to $(v_3, \sigma(v_3))$ but not to $(v_2, \sigma(v_2))$. Thus it can be coloured using the same colour as for $(v_2, \sigma(v_2))$. Proceeding like this, the S-chromatic polynomial of the S-path P_n^S, will be given by:

$$M_{P_n^S}(k) = \left(\prod_{i=1}^{n} \sigma(v_i), k^{\left\lceil \frac{n}{2} \right\rceil} (k-1)^{\left\lceil \frac{n}{2} \right\rceil} \right)$$

Theorem 3.4

For any even S-cycle C_n^S, the S-chromatic polynomial is given by,

$$M_{C_n^S}(k) = \left(\prod_{i=1}^{n} \sigma(v_i), k^{\left\lceil \frac{n}{2} \right\rceil} (k-1)^{\left\lceil \frac{n}{2} \right\rceil} \right)$$

Proof

As in the case of *S*-path vertices can be coloured alternatively using any of the *k*-colours for one vertex and any of the *(k-1)* colours from C $\setminus \{c_1\}$. Thus in the case of even cycle the vertices can be coloured using only two different colours. Hence we obtain the results

$$M_{C_n^S}(k) = \left(\prod_{i=1}^{n} \sigma(v_i), k^{\left\lceil \frac{n}{2} \right\rceil} (k-1)^{\left\lceil \frac{n}{2} \right\rceil} \right)$$

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Theorem 3.5

For any odd S-cycle C_n^S, the vertex S-chromatic polynomial is given by,

$$M_{C_n^S}(k)=\prod_{i=1}^{n}\left(\sigma(v_i),k^{\left\lfloor\frac{n}{2}\right\rfloor}(k-1)^{\left\lfloor\frac{n}{2}\right\rfloor}(k-2)^{n-2\left\lfloor\frac{n}{2}\right\rfloor}\right)$$

Proof:

Consider the odd *S*-cycle C_n^S with the n vertices $(v_1,\sigma(v_1)),(v_2,\sigma(v_2)),\cdots,(v_n,\sigma(v_n))$. Since *n* is odd (*n-1*) is an even integer. Therefore we can colour the vertices $(v_1,\sigma(v_1)),(v_2,\sigma(v_2)),\cdots,(v_{n-1},\sigma(v_{n-1}))$ alternatively by using, any one of the k-colours, say c_1, and any one of the $(k-1)$ colours, say c_2, from C $\setminus\{c_1\}$. Now the vertex $(v_n,\sigma(v_n))$ is adjacent to the vertices $(v_1,\sigma(v_1))$ with colour c_1 and the vertex $(v_{n-1},\sigma(v_{n-1}))$ with colour c_2. Hence the vertex $(v_n,\sigma(v_n))$ cannot be assigned the colours c_1 and c_2. Hence it should be assigned any one of the $(k-2)$ colours from C $\setminus\{c_1,c_2\}$. Thus the vertex S-chromatic polynomial of the S- cycle C_n^S is given by

$$M_{C_n^S}(k)=\prod_{i=1}^{n}\left(\sigma(v_i),k^{\left\lfloor\frac{n}{2}\right\rfloor}(k-1)^{\left\lfloor\frac{n}{2}\right\rfloor}(k-2)^{n-2\left\lfloor\frac{n}{2}\right\rfloor}\right)$$

Theorem 3.6

For any S-complete graph K_n^S, the vertex S-chromatic polynomial is given by

$$M_{K_n^S}(k)=\left(\prod_{i=1}^{n}\sigma(v_i),(k(k-1)(k-2)...(k-(n-1)))\right)\ k\geq n$$

Proof

Consider the *S*-complete graph K_n^S. Let V be the set of vertices set $(v_1,\sigma(v_1)),(v_2,\sigma(v_2)),\cdots,(v_n,\sigma(v_n))$, and let C $=\{c_1,c_2,\cdots,c_n\}$ be the different types of colours. In K_n^S each vertex $(v_i,\sigma(v_i)),i=1,2,\cdots n$ is adjacent to every other vertex $(v_j,\sigma(v_j)),\ j\neq i,\ j=1,2,\cdots n$. Hence if $(v_i,\sigma(v_i)),$ is assigned any of the k colours say c_1 then each other vertex $(v_j,\sigma(v_j)),\ j\neq i$ should be coloured using any one of the $(k-1)$, $(k-2),\cdots k-(n-1)$ colours respectively. Thus the vertex S-chromatic polynomial of K_n^S is given by,

$$M_{K_n^S}(k)=\left(\prod_{i=1}^{n}\sigma(v_i),(k(k-1)(k-2)...(k-(n-1)))\right)\ k\geq n$$

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Theorem 3.7

For any S-complete bipartite graph $K_{m,n}^S$,

$$M_{K_{m,n}^S} = \left(\prod_{i=1}^{n} \sigma(v_i), k^m (k-1)^n \right)$$

Proof

In a complete bipartite graph $K_{m,n}^S$ the set of vertices can be partition into two subsets V_1 of *m* vertices and V_2 of *n* vertices. Each vertex $(v_{1i}, \sigma(v_{1i})), i = 1,2,\cdots,n$ is adjacent to every vertex $(v_{2j}, \sigma(v_{2j})), j = 1,2,\cdots,n$ however $(v_{1i}, \sigma(v_{1i}))$ and $(v_{1k}, \sigma(v_{1k}))$, $k \neq i, k = 1,2,\cdots,n$ are not adjacent in the set V_1. Similarly the vertices $(v_{2j}, \sigma(v_{2j}))$, $j = 1,2,\cdots,n$ and $(v_{2l}, \sigma(v_{2l})), l \neq j, l = 1,2,\cdots,n$ are not adjacent in V_2. Thus each vertex $(v_{1i}, \sigma(v_{1i}))$ in V_1 can be coloured by using the same colour out of the *k*- colours available in the colour class C of *k*-colours. Similarly each vertex $(v_{2j}, \sigma(v_{2j}))$ can be assigned the same colour among the *(k-1)* colours available in C Hence the vertex S-chromatic polynomial of the S-complete bipartite graph $K_{m,n}^S$ is given by,

$$M_{K_{m,n}^S} = \left(\prod_{i=1}^{n} \sigma(v_i), k^m (k-1)^n \right)$$

In the above proof if we choose *m=1* we obtain the following corollary.

Corollary 3.8

For any S-star $K_{1,n}^S$, the S-chromatic polynomial is given by,

$$M_{K_{1,n}^S}(k) = \left(\prod_{i=1}^{n} \sigma(v_i), k(k-1)^n \right)$$

Combining theorems 3.4, 3.5 and corollary 3.8, we obtain the following results for a S-wheel.

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Theorem 3.9

For any even S-wheel, the vertex S-chromatic polynomial is given by,

$$M_{W_n^S}(k)=\begin{cases}\left(\prod_{i=1}^{n}\sigma(v_i), k(k-1)^{\left\lceil\frac{n-1}{2}\right\rceil}(k-2)^{\left\lceil\frac{n-1}{2}\right\rceil}\right) & if \quad n\, is\, even\\ \left(\prod_{i=1}^{n}\sigma(v_i), k(k-1)^{\left\lceil\frac{n-1}{2}\right\rceil}(k-2)^{\left\lceil\frac{n-1}{2}\right\rceil}(k-3)^{n-\left(2\left\lceil\frac{n-1}{2}\right\rceil\right)}\right) & if \quad n\, is\, odd\end{cases}$$

4. CONCLUSION

This paper introduces the concept of vertex chromatic polynomial for S-valued graphs. In future, further type of chromatic polynomial will be discussed.

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